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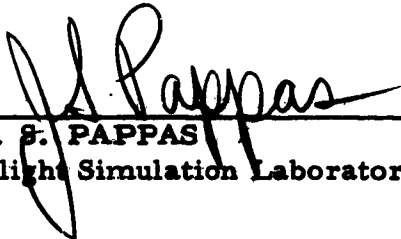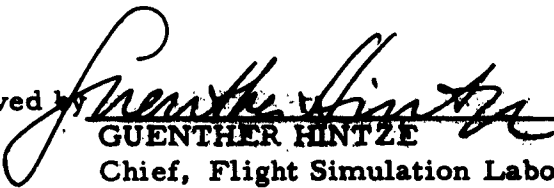
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INTERNAL MEMORANDUM
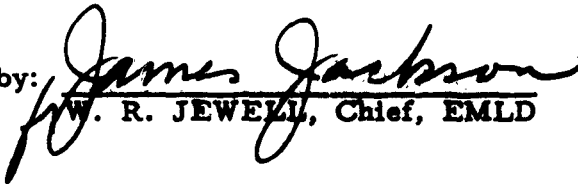
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NON-LINEAR MATRIX RICCATI RELATIONS OCCURING IN FLIGHT DYNAMICS

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ELECTRO-MECHANICAL LABORATORIES DIVISION

INTERNAL MEMORANDUM

Nr. 3

NON-LINEAR MATRIX RICCATI RELATIONS OCCURING IN FLIGHT DYNAMICS

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ABSTRACT

This report shows systematic techniques for handling or "packaging" of n-tuples of n-tuples of position, velocity, and acceleration vectors. Large systems of vectors cast into state-vector representations lead, in a natural way, to matrix Riccati Equations.

TABLE OF CONTENTS

INTRODUCTION

The scalar Riccati Equation as given by Ince (Reference 1) is

$$\dot{y} + ay^2 + by + c = 0 \quad (1\text{-}1)$$

where y is a scalar-valued variable of the scalar argument (variable) time, a, b, and c are constants or time varying coefficients.

The matrix Riccati Equation as given by Levine (Reference 2) is

$$\dot{Y} + B_1 Y + Y B_2 + Y' A Y + C = [0] \quad (1\text{-}2)$$

where Y (t) is a matrix-valued function of a scalar argument (single variable time) and A, B_1, B_2 and C are constant or time-varying coefficient-matrices.

Kalman, Bellman, Pontrjagin and other researchers working in the area of mathematical programming and optimization frequently refer to the matrix Riccati Equation. The determination of an optimal control in feedback control theory or of an optimal filter in estimation and prediction theory invokes a consideration of the Riccati Equation.

This report shows how numerously one can obtain matrix Riccati relations in vector dynamics. A geometrical insight into the relations described by the matrices is presented.

Section I. Some Aspects of Generalized Dynamical Systems

This section considers systems of vectors and their first and second derivatives. Representations of the vectors independent of their coordinates in bases are presented parallel to matrices of their coordinates relative to moving bases. Matrices of invariant inner-products of velocity vectors, representing a matrix of scalars of the motion of a system of vectors, motivates the notion of generalized energies. A scalar invariant of the system is generated as the trace of the matrix and has its analog in total system energy.

Many other classical mechanics notions are presented in a unifying systematic manner based on fundamentals of mathematical transformation theory.

Consider a vector $\bar{x}$ in real n-space with coordinates in a fixed basis $\overleftarrow{f}$ and in two other arbitrary bases $\overleftarrow{y}$ and $\overleftarrow{z}$, that is

$$\bar{x} = \left\langle \overleftarrow{f} \ x^{f} \right\rangle = \left\langle \overleftarrow{y} \ x^{y} \right\rangle = \left\langle \overleftarrow{z} \ x^{z} \right\rangle . \tag{1}$$

The coordinates of the $\overleftarrow{y}$ basis in the $\overleftarrow{f}$ space is

$$\left\langle \overleftarrow{y} = \left\langle \overleftarrow{f} \ Y^{f} \right. \right. \tag{2}$$

and similiarly

$$\left\langle \overleftarrow{z} = \left\langle \overleftarrow{f} \ Z^{f} \right. \right. \tag{3}$$

The system of 2n + 1 vectors

$$\left\langle \overleftarrow{s} = (\bar{x}, \left\langle \overleftarrow{y}, \left\langle \overleftarrow{z} \right) = (\bar{x}, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{z}_1, \bar{z}_2, \bar{z}_3) \right. \right. \tag{4}$$

in n-space with no constraints represents n(2n + 1) independent coordinates that is $2n^2 + n$ scalars (field elements) specify the configuration of the seven vectors. Each second-derivative vector (e.g. $\ddot{\bar{x}}$ in n-space requires 2 n integrators (with their initial conditions) to obtain the vector.

2

Hence the dynamical propagation of the system of vectors

$$\langle \ddot{\bar{q}} = (\ddot{\bar{x}}, \langle \ddot{\bar{y}}, \langle \ddot{\bar{z}}) = (\ddot{\bar{x}}, \ddot{\bar{y}}_1, \ddot{\bar{y}}_2, \ddot{\bar{y}}_3, \ddot{\bar{z}}_1, \ddot{\bar{z}}_2, \ddot{\bar{z}}_3) \qquad (5)$$

requires $2 \times n \times 7 = 14 n$ integrators. In three space one has 42 integrators in a computer mechanization of the system. The time derivatives of (5) are with respect to the fixed background basis, that is

$$\langle \dot{\bar{f}} = \langle \bar{0} . \qquad (6)$$

Using (2) and (3) in (1), one obtains

$$x^f \rangle = Y^f \; x^y \rangle = Z^f \; x^z \rangle \qquad (7)$$

hence for complete flexibility in mapping between bases, the transformation matrices Y and Z, must be obtained. These transformations as well as a number of other relations are obtained as solutions of matrix Riccati type equations.

Before bogging down in the detailed maze of matrices of coordinates, let us glimpse at the overall second-derivative picture of the system of vectors. Note that above we state matrices of coordinates. This handling of large packages of coordinates (matrices) alleviates the tediousness of the standard classical treatise of mechanics systems of particles wherein one deals with a maze of super and subscripts which rapidly swamp the meaning of the physical picture.

A. **Matrices of Invariant Representations**

The relative configuration of the system of vectors may be described by a matrix of inner-products (independent of coordinates) by

3

$$\vec{s}\rangle \cdot \langle \bar{s} = \begin{pmatrix} \bar{x} \\ \vec{y}\rangle \\ \vec{z}\rangle \end{pmatrix} \cdot (\bar{x}, \langle \bar{y}, \langle \bar{z}) = Q_{ss} \tag{8}$$

or

$$Q_{ss} = \begin{pmatrix} \bar{x} \cdot \bar{x} & \bar{x} \cdot \langle \bar{y} & \bar{x} \cdot \langle \bar{z} \\ \vec{y}\rangle \cdot \bar{x} & \vec{y}\rangle \cdot \langle \bar{y} & \vec{y}\rangle \cdot \langle \bar{z} \\ \vec{z}\rangle \cdot \bar{x} & \vec{z}\rangle \cdot \langle \bar{y} & \vec{z}\rangle \cdot \langle \bar{z} \end{pmatrix} \tag{9}$$

where geometrically one has the conventional notion of

$$\bar{x} \cdot \bar{x} = |\bar{x}| \; |\bar{x}| \tag{10}$$

$$\vec{y}\rangle \cdot \bar{x} = (\bar{y}_1 \cdot \bar{x}, \; \bar{y}_2 \cdot \bar{x}, \; \bar{y}_3 \cdot \bar{x}) \tag{11}$$

$$\vec{y}\rangle \cdot \langle \bar{z} = \begin{bmatrix} \bar{y}_1 \cdot \bar{z}_1 & \bar{y}_1 \cdot \bar{z}_2 & \bar{y}_1 \cdot \bar{z}_3 \\ \bar{y}_2 \cdot \bar{z}_1 & \bar{y}_2 \cdot \bar{z}_2 & \bar{y}_2 \cdot \bar{z}_3 \\ \bar{y}_3 \cdot \bar{z}_1 & \bar{y}_3 \cdot \bar{z}_2 & \bar{y}_3 \cdot \bar{z}_3 \end{bmatrix} = Q_{yz} \; , \tag{12}$$

One may now write (9) as

$$Q_{ss} = \begin{pmatrix} \bar{x} \cdot \bar{x} & \bar{x} \cdot \langle \bar{y} & \bar{x} \cdot \langle \bar{z} \\ \vec{y}\rangle \cdot \bar{x} & Q_{yy} & Q_{yz} \\ \vec{z}\rangle \cdot \bar{x} & Q_{zy} & Q_{zz} \end{pmatrix} . \tag{13}$$

The first derivative dynamics of the system is described by

$$\dot{Q}_{ss} = \dot{\bar{s}}\rangle \cdot \langle \bar{s} + \bar{s}\rangle \cdot \langle \dot{\bar{s}} \tag{14}$$

or

4

$$Q_{ss} = \begin{pmatrix} \dot{\vec{x}} \\ \dot{\vec{y}} \\ \dot{\vec{z}} \end{pmatrix} \cdot (\vec{x}, <\vec{y}, <\vec{z}) + \begin{pmatrix} \dot{\vec{x}} \\ \dot{\vec{y}} \\ \dot{\vec{z}} \end{pmatrix} \cdot (\vec{x}, <\dot{\vec{y}}, <\dot{\vec{z}}) \qquad (15)$$

Note that the time derivative terms of (15) are coordinate free. In later sections the matrices of coordinates will enter the picture.

The second derivative dynamics of the system is described by

$$\ddot{Q}_{ss} = \ddot{\vec{s}}> \cdot <\vec{s} + 2 \dot{\vec{s}}> \cdot <\dot{\vec{s}} + \vec{s}> \cdot <\ddot{\vec{s}} \qquad (16)$$

Clearly one detects the notion of total system kinetic energy or rather a matrix of some of the scalars of the motion in the matrix of inner-products $\dot{\vec{s}}> \cdot <\dot{\vec{s}}$, that is

$$\dot{\vec{s}}> \cdot <\dot{\vec{s}} = \begin{pmatrix} \dot{\vec{x}} \cdot \dot{\vec{x}} & \dot{\vec{x}} \cdot <\dot{\vec{y}} & \dot{\vec{x}} \cdot <\dot{\vec{z}} \\ \dot{\vec{y}}> \cdot \dot{\vec{x}} & \dot{\vec{y}}> \cdot <\dot{\vec{y}} & \dot{\vec{y}}> \cdot <\dot{\vec{z}} \\ \dot{\vec{z}}> \cdot \dot{\vec{x}} & \dot{\vec{z}}> \cdot <\dot{\vec{y}} & \dot{\vec{z}}> \cdot <\dot{\vec{z}} \end{pmatrix} \qquad (17)$$

The total "energy" is the trace of (17), that is

$$e = <\dot{\vec{s}} \cdot \dot{\vec{s}}> = \text{trace } \dot{\vec{s}}> \cdot <\dot{\vec{s}} \qquad (18)$$

or

$$<\dot{\vec{s}} \cdot \dot{\vec{s}}> = \dot{\vec{x}} \cdot \dot{\vec{x}} + \text{trace } \dot{\vec{y}}> \cdot <\dot{\vec{y}} + \text{trace } \dot{\vec{z}}> \cdot <\dot{\vec{z}}$$

$$e = \dot{\vec{x}} \cdot \dot{\vec{x}} + <\dot{\vec{y}} \cdot \dot{\vec{y}}> + <\dot{\vec{z}} \cdot \dot{\vec{z}}> . \qquad (19)$$

The trace of a matrix is an invariant (independent of bases.)

The analogs of the classical concepts of energy, power, etc, for a system of vectors may also be formulated and handled at the matrix of vectors level. Lator reports will present these generalized systems of vectors in generalized coordinates in a Hamiltonian framework.

5

B. **Matrices of Matrices of Coordinates**

Consider the matrix of coordinates of the $\overleftarrow{z}$ vectors in the $\overleftarrow{\bar{y}}$ basis, that is by (2) and (3)

$$\overleftarrow{\bar{z}} = \overleftarrow{\bar{y}}\, Y^{-1} z = \overleftarrow{\bar{y}}\, T \; . \tag{20}$$

The system of vectors $\overleftarrow{\bar{s}}$ is now

$$\overleftarrow{\bar{s}} = (\bar{x}, \overleftarrow{\bar{y}}, \overleftarrow{\bar{y}}\, T) \tag{21}$$

and

$$\overrightarrow{\bar{s}} \cdot \overleftarrow{\bar{s}} = \begin{bmatrix} \bar{x}.\bar{x} & \bar{x}\cdot\overleftarrow{\bar{y}} & \bar{x}\cdot\overleftarrow{\bar{y}}\,T \\ \overrightarrow{\bar{y}}\cdot\bar{x} & M_{yy} & M_{yy}\,T \\ T'\overrightarrow{\bar{y}}\cdot\bar{x} & T'M_{yy} & T'M_{yy}\,T \end{bmatrix} \tag{22}$$

The matrix of scalars T of (20) may be evaluated as a matrix of inner-products by projecting the $\overleftarrow{\bar{z}}$ vectors onto the dual vectors $\overrightarrow{\bar{y}}$, that is

$$\overset{*}{\overrightarrow{\bar{y}}}\cdot\overleftarrow{\bar{z}} = \begin{bmatrix} \bar{y}_i \cdot \bar{z}_j \end{bmatrix} = \overset{*}{\overrightarrow{\bar{y}}}\cdot\overleftarrow{\bar{y}}\,T = T \; , \tag{23}$$

since by definition of duals or reciprocals

$$\overset{*}{\overrightarrow{\bar{y}}}\cdot\overleftarrow{\bar{y}} = I \; . \tag{24}$$

It is shown in later sections that the matrix T satisfies a matrix Riccati differential equation.

C. **The Alias - Alibi Aspect**

In algebraic transformation theory one is confronted with the vector matrix equation

$$\overrightarrow{w} = A \overrightarrow{u} \tag{25}$$

6

which has two interpretations:

(1) $u\rangle$ and $v\rangle$ are two different vectors.

(2) $u\rangle$ and $v\rangle$ are the same vectors in different coordinate systems (bases). Clearly (25) is the same form as (7).

Halmos[*]points out that the theory of changing bases is coextensive with the theory of invertible linear transformations. He states that an invertible linear transformation is an automorphism, whereby an automorphism means an isomorphism of a vector space with itself. Halmos points out that conversely, every automorphism is an invertible linear transformation.

The study of the properties of a linear transformation A on a vector $u\rangle$ and of the structure of the transformation A necessitates that one study the effect of A on n-linearly independent vectors $u\rangle_1, u\rangle_2, \ldots u\rangle_n$.

For example, in the three space three linearly independent vectors $u\rangle_i$ in the domain space under A map to three vectors in the Range space with respect to A as

$$\left[v\rangle_1 , v\rangle_2 , v\rangle_3 \right] = \left[A\, u\rangle_1, A\, u\rangle_2, A\, u\rangle_3 \right] \quad (26)$$

or

$$W = A\,U \quad (27)$$

Since U are linearly independent their matrix of coordinates U is an invertible matrix, hence

$$A = W\,U^{-1} . \quad (28)$$

*Halmos, P. 86.

7

Clearly, then in the study of the dynamics of the single vector $\bar{x}$, one must study the motion of n linearly independent vectors

$$\langle \bar{x} = (\bar{x}_1, \bar{x}_2 \ldots \bar{x}_n) \tag{29}$$

In differential equation theory this is equivalent to obtaining n linearly independent solutions.

For example, if the vector $\bar{x}$ is known to propagate under a linear law A(t) where A is a

$$\dot{x}\rangle = A(t)x\rangle \tag{30}$$

function of time, one must consider the extended system of (4) as

$$\langle \bar{s} = (\langle \bar{x}, \langle \bar{y}, \langle \bar{z}) \tag{31}$$

to obtain the dynamics of $\bar{x}$ in two arbitrary sets of moving bases $\langle \bar{y}$ and $\langle \bar{z}$. In other words, one must consider n-linearly independent solution vectors.

This paper will not go into the study of the analytical properties of the systems of dynamical vectors of (31).

Section II. Vector System Equations of Motion

This report is limited to formulation of the equation of motion for the system of (I-4) for computer mechanization and to yeild geometrical insight into the geometrical dynamics.

A. First Derivative Vector Matrix Relations

The vector $\bar{x}$ by (I-1) in the two arbitrary moving bases is

$$\bar{x} = \left\langle \bar{y}\ x^y \right\rangle = \left\langle \bar{z}\ x^z \right\rangle = \tag{1}$$

and the derivative with respect to time is

$$\dot{\bar{x}} = \left\langle \dot{\bar{y}}\ x^y \right\rangle + \left\langle \bar{y}\ \dot{x}^y \right\rangle = \left\langle \dot{\bar{z}}\ x^z \right\rangle + \left\langle \bar{z}\ \dot{x}^z \right\rangle . \tag{2}$$

Since $\left\langle \bar{y} \right.$ is a basis, the three velocity vectors $\left\langle \dot{\bar{y}} \right.$ may be exp.essed in the $\left\langle \bar{y} \right.$ basis, that is a matrix of scalars exist such that

$$\left\langle \dot{\bar{y}} \right. = \left\langle \bar{y}\right.\ V\ (y) \tag{3}$$

and similarly for the $\left\langle \bar{z} \right.$ vectors

$$\left\langle \dot{\bar{z}} \right. = \left\langle \bar{z}\right.\ V\ (z) \tag{4}$$

By (I-2)

$$\left\langle \bar{y} \right. = \left\langle \bar{r}\right.\ Y \tag{5}$$

and

$$\left\langle \bar{r} \right. = \left\langle \bar{y}\right.\ Y^{-1} \tag{6}$$

hence

$$\left\langle \dot{\bar{y}} \right. = \left\langle \bar{r}\right.\ \dot{Y} \tag{7}$$

since by (I-6)

$$\langle \dot{\bar{f}} = \langle \bar{0} \tag{8}$$

hence

$$\langle \dot{\bar{y}} = \langle \bar{y} \, Y^{-1} \dot{Y} \, . \tag{9}$$

By (3) and (9)

$$Y^{-1} \dot{Y} = V(y) \, , \tag{10}$$

and similarly for the basis $\langle \bar{z}$, that is

$$\langle \dot{\bar{z}} = \langle \bar{z} \, Z^{-1} \dot{Z} = \langle \bar{z} \, V(z) \tag{11}$$

By (10) and (11)

$$\boxed{\dot{Y} = Y \, V(y)} \tag{12}$$

$$\boxed{\dot{Z} = Z \, V(z)} \, . \tag{13}$$

The two first order matrix differential equations above are special cases of the Riccati matrix equation shown in(1-2).

By (I-20)

$$\langle \bar{z} = \langle \bar{y} \, T = \langle \bar{y} \, Y^{-1} \, z \tag{14}$$

hence also

$$\langle \dot{\bar{z}} = \langle \dot{\bar{y}} \, T + \langle \bar{y} \, \dot{T} \tag{15}$$

10

Using (9) and (11) in (15)

$$\langle \bar{y} \; \dot{T} = \langle \bar{y} \; T \; V \; (z) + \langle \bar{y} \; V \; (y) \; T \tag{16}$$

or

$$\boxed{\dot{T} = T \; V \; (z) + V \; (y) \; T} \tag{17}$$

which is also a matrix Riccati equation as seen by (1-2).

One may derive (17) directly from the matrix relations of (14), that is

$$T = Y^{-1} \; Z \tag{18}$$

and

$$\dot{T} = \left(\frac{d}{dt} Y^{-1} \right) Z + Y^{-1} \; \dot{Z} \; . \tag{19}$$

now

$$Y^{-1} \; Y = I \tag{20}$$

hence

$$\frac{d}{dt} Y^{-1} \; (Y) + Y^{-1} \; \dot{Y} = [0] \tag{21}$$

therefore

$$\frac{d}{dt} (Y^{-1}) = Y^{-1} \; \dot{Y} \; Y^{-1} \tag{22}$$

Equation (22) in (19) yields

$$\dot{T} = Y^{-1} \; \dot{Y} \; Y^{-1} \; Z + Y^{-1} \; \dot{Z} \tag{23}$$

11

Using the identity

$$Z Z^{-1} = I \tag{24}$$

(23) becomes

$$\dot{T} = Y^{-1} \dot{Y} (Y^{-1} Z) + (Y^{-1} Z) Z^{-1} \dot{Z} \tag{25}$$

Using (18) in (25)

$$\dot{T} = Y^{-1} \dot{Y} T + T Z^{-1} \dot{Z} \tag{26}$$

Clearly if Y and Z satisfy

$$\dot{Y} = Y V(y) \tag{27}$$

and

$$\dot{Z} = Z V(z) \tag{28}$$

then (26) may be written as

$$\dot{T} = V(y) T + T V(z) \tag{29}$$

which agrees with (17).

The latter derivation is mechanical manipulation and does not convey the direct geometrical clarity as does the first derivation leading to (17).

Consider the metric-matrix of the $\overleftarrow{\overline{y}}$ basis M_{yy}, that is

$$M_{yy} = \overline{y} \rangle \cdot \langle \overline{y} = Y' M_{ff} Y \tag{30}$$

12

The derivative of (30) is

$$\dot{M}_{yy} = \dot{\bar{y}} \rangle \cdot \langle \bar{y} + \bar{y} \rangle \cdot \langle \dot{\bar{y}} \quad . \tag{31}$$

By (3)

$$\langle \dot{\bar{y}} = \langle \bar{y} \, V \, (y) \tag{32}$$

and transposing

$$\dot{\bar{y}} \rangle = V'(y) \, \bar{y} \rangle \tag{33}$$

hence using (32) and (33) in (31) yields

$$\boxed{\dot{M}_{yy} = V'(y) \, M_{yy} + M_{yy} \, V \, (y)} \tag{34}$$

which once again is a special case of the Matrix Riccati equation (1-2).

A repetition of the same argument for the dynamics of the $\bar{z}$ basis yields

$$\boxed{\dot{M}_{zz} = V'(z) \, M_{zz} + M_{zz} \, V \, (z)} \, . \tag{35}$$

Consider next the matrix of inner-products between the bases $\langle \bar{y}$ and $\langle \bar{z}$, that is

$$Q = \bar{y} \rangle \cdot \langle \bar{z} = M_{yy} \, T \tag{36}$$

The matrix Q specifies the relative orientations of the vectors. The derivative of (36) is

$$\dot{Q} = \dot{\bar{y}} \rangle \cdot \langle \bar{z} + \bar{y} \rangle \cdot \langle \dot{\bar{z}} \tag{37}$$

13

Using (33) and (11) in (37)

$$\dot{Q} = V'(y) \, \vec{y} \rangle \cdot \langle \bar{z} + \vec{y} \rangle \cdot \langle \bar{z} V \, (z) \tag{38}$$

and by (36)

$$\boxed{\dot{Q} = V'(y) \, Q + Q \, V \, (z)} \tag{39}$$

B. **Second Derivative Vector-Matrix Relations**

The vector $\bar{x}$ in the $\langle \vec{r}, \langle \bar{y}$ and the $\langle \bar{z}$ bases by (I-1) is

$$\bar{x} = \langle \vec{r} \, x^r \rangle = \langle \bar{y} \, x^y \rangle = \langle \bar{z} \, x^z \rangle \tag{40}$$

and the velocity vector is

$$\dot{\bar{x}} = \langle \vec{r} \, \dot{x}^r \rangle = \langle \dot{\bar{y}} \, x^y \rangle + \langle \bar{y} \, \dot{x}^y \rangle \tag{41}$$

$$= \langle \dot{\bar{z}} \, x^z \rangle + \langle \bar{z} \, \dot{x}^z \rangle \quad . \tag{42}$$

The acceleration vector is

$$\ddot{\bar{x}} = \langle \vec{r} \, \ddot{x}^r \rangle = \langle \ddot{\bar{y}} \, x^y \rangle + 2 \langle \dot{\bar{y}} \, \dot{x}^y \rangle + \langle \bar{y} \, \ddot{x}^y \rangle \tag{43}$$

$$= \langle \ddot{\bar{z}} \, x^z \rangle + 2 \langle \dot{\bar{z}} \, \dot{x}^z \rangle + \langle \bar{z} \, \ddot{x}^z \rangle \quad . \tag{44}$$

14

Section III. <u>Trajectory of a Vector $\bar{x}$ and one Arbitrary Time Varying Basis</u> $\langle \bar{y}$.

The first case will consider the simple system of four arbitrary vectors with the only condition that the three vectors $\langle \bar{y}$ form a basis, that is

$$\langle \bar{s} = (\bar{x}, \langle \bar{y}) = (\bar{x}(t), \bar{y}_1(t), \bar{y}_2(t), \bar{y}_3(t)) \qquad (45)$$

where

$$\bar{x} = \left\langle \bar{f}\, x^f(t) \right\rangle = \left\langle \bar{y}\, x^y \right\rangle \qquad (46)$$

and

$$\langle \bar{y} = \langle \bar{f}\, Y(t) . \qquad (47)$$

In three space we have a system of four vectors each having three independent variables or a total of twelve arbitrary coordinates to specify the geometrical configuration of the system of vectors.

Clearly, at the velocity level we need twelve integrations and

$$\langle \dot{\bar{s}} = \left\langle \bar{f} \left[\underset{3\times 1}{\left\langle \dot{x}^f \right\rangle} , \underset{3\times 3}{\dot{Y}(t)} \right] = \left\langle \bar{v} \right. \qquad (48)$$

$$3\times 4$$

twelve initial conditions or constants of the motion as shown by (48) where the components of the 3 x 4 matrix are all in the "fixed" bases $\langle \bar{f}$.

In most real-life problems the parameters of the system and the constants of the system occur most naturally in moving bases.

Clearly, the second derivative vectors of the system require 24 integrations, that is

$$\left\langle \overset{\cdot\cdot}{s} = \left\langle \overset{\cdot}{?} \left[\overset{\cdot\cdot}{x^{\gamma}} \right\rangle , \overset{\cdot\cdot}{Y}(t) \right] = \left\langle \overset{\cdot}{v} \right. \tag{49}$$

and 24 initál conditions, the additional twelve are conditions on initial velocities, since

$$\overset{\cdot}{x}\!\left.\right\rangle = v(x)\!\left.\right\rangle$$

and

$$\overset{\cdot}{Y} = V(y)$$

hence

$$\left\langle \overset{\cdot\cdot}{s} = \left\langle \overset{\cdot}{?} \left[\overset{\cdot}{v}(x)\!\left.\right\rangle , \overset{\cdot}{v}(y) \right] \right. \tag{50}$$

The acceleration vector $\overset{\cdot\cdot}{x}$ in the moving basis $\left\langle \bar{y} \right.$ by (43) is

$$\overset{\cdot\cdot}{x} = \left\langle \overset{\cdot\cdot}{\bar{y}} \, x^{\gamma} \right\rangle + 2 \left\langle \overset{\cdot}{\bar{y}} \, \overset{\cdot}{x^{\gamma}} \right\rangle + \left\langle \bar{y} \, \overset{\cdot\cdot}{x^{\gamma}} \right\rangle . \tag{51}$$

By (32)

$$\left\langle \overset{\cdot}{\bar{y}} = \left\langle \bar{y} \, V(y) \right. \tag{52}$$

and

$$\left\langle \overset{\cdot\cdot}{\bar{y}} = \left\langle \overset{\cdot}{\bar{y}} \, V(y) + \left\langle \bar{y} \, \overset{\cdot}{V}(y) \right. \tag{53}$$

or

$$\left\langle \overset{\cdot\cdot}{\bar{y}} = \left\langle \bar{y} \left[V^{2}(y) + \overset{\cdot}{V}(y) \right] \right. \tag{54}$$

16

Using (52) and (53) in (50)

$$\ddot{\bar{x}} = \left\langle \bar{y} \left\{ \left. \ddot{x} \right\rangle^y + 2\ V(y)\ \left. \dot{x} \right\rangle^y + \left[V^2(y) + \dot{V}(y) \right] \left. x \right\rangle^y \right\} \right. . \quad (55)$$

also

$$\ddot{\bar{x}} = \bar{a} = \left\langle \bar{y}\ a \right\rangle^y \quad (56)$$

where $\bar{a}$ may be considered the driving vector (specific force in mechanics), hence

$$\left. \ddot{x} \right\rangle^y + 2\ V(y)\ \left. \dot{x} \right\rangle^y + (V^2 + \dot{V})\ \left. x \right\rangle^y = \left. a \right\rangle^y \quad (57)$$

Equation (57) is a second-order matrix differential equation linear in the vector variable $x\rangle^y$ and its derivatives, that is

$$\left. \ddot{x} \right\rangle + A_2(t)\ \left. \dot{x} \right\rangle + A_1(t)\ \left. x \right\rangle = a \right\rangle \quad (58)$$

where in general the vector $a\rangle$ is a non-linear function of $x\rangle$ and $\dot{x}\rangle$, that is

$$a\rangle = a\left(x\rangle, \dot{x}\rangle \right)\rangle \quad (59)$$

depending on the nature of $a\rangle$.

It has been observed that we need 24 integrations for the systems under consideration. Six integrations are required for (58), however, the position and the velocity coefficient matrices A_1 and A_2 must be determined. Observe that one vector $\bar{x}$ requires six integrators and that the system of three vectors $\langle \bar{y}$ will require 18 more integrations.

By equation (54)

$$\langle \ddot{\bar{y}} = \langle \bar{y} \left[V^2(y) + \dot{V}(y) \right] = \langle \bar{y} \, H \tag{60}$$

that is

$$\left(\ddot{\bar{y}}_1, \ddot{\bar{y}}_2, \ddot{\bar{y}}_3 \right) = \left(\bar{h}_1, \bar{h}_2, \bar{h}_3 \right) \tag{61}$$

and the three "driving" vectors $\langle \bar{h}$ have coordinates in the $\langle \bar{y}$ basis as

$$\langle \bar{h} = \langle \bar{y} \, H \quad . \tag{62}$$

Thus, if the matrix H is known the first order non-linear matrix equation can be mechanized, that is

$$\dot{V}(y) + V^2(y) = H \, . \tag{63}$$

Equation (63) requires nine integrations and nine initial conditions. Equation (27) provides the other nine integrations, that is

$$\dot{Y} = Y \, V(y) \quad . \tag{64}$$

The system of vector and matrix differential equations is

$$\boxed{\begin{aligned} & \ddot{x\bar{y}}\rangle + 2 \, V(y) \, \dot{x\bar{y}}\rangle + \left[V^2(y) + \dot{V}(y) \right] x\bar{y}\rangle = a\bar{y}\rangle \\ & \qquad\qquad\qquad \text{6 integrators} \\ & \dot{V}(y) + V^2(y) = H \\ & \qquad\qquad\qquad \text{9 integrators} \\ & \dot{Y}(f) + Y \, V = [0] \\ & \qquad\qquad\qquad \text{9 integrators} \end{aligned}} \tag{65}$$

If the constraints are imposed on the basis $\langle \bar{y}$, that they are orthonormal, that is, that

$$M_{yy} = \vec{\bar{y}} \rangle \cdot \langle \bar{y} = I = \left[\bar{y}_i \cdot \bar{y}_j \right] \tag{66}$$

then the symmetric matrix contains six independent conditions to be satisfied. These conditions are

$$\bar{y}_i \cdot \bar{y}_i = 1 \quad i = 1,2,3 \tag{67}$$

which is three conditions, and

$$\bar{y}_i \cdot \bar{y}_j = 0 \quad i \neq j \tag{68}$$

since

$$y_i \cdot y_j = y_j \cdot y_i \tag{69}$$

the six off-diagonal conditions are redundant because of symmetry, that is only three independent conditions.

From the above discussion, it is clear that the three vectors $\langle \bar{y}$ with the constraint (66) are completely specified by three independent coordinates instead of nine, that is the nine elements of the matrix Y are functions of only three variables $q \rangle$, hence

$$\langle \bar{y} = \langle \bar{f} \, Y \, (q_1, q_2, q_3) \quad . \tag{70}$$

It is well known that when $\langle \bar{f}$ is also on O.N basis, or

$$M_{ff} = I \tag{71}$$

$$\vec{\bar{y}} \rangle = Y' \vec{\bar{f}} \rangle \tag{72}$$

19

hence

$$\vec{y}\rangle \cdot \langle\bar{y} = Y' \, M_{ff} \, Y \quad . \tag{73}$$

By (66) and (71)

$$I = Y'Y \tag{74}$$

hence

$$Y' = Y^{-1}$$

and Y is said to be an orthogonal matrix.

The theory of Euler angles and relations to O.N bases is well known, that is

$$\langle\bar{y} = \langle\bar{r}\, M_k(\phi_3)\, M_j(\phi_2)\, M_i(\phi_1) = \langle\bar{r}M(\phi\rangle)$$

where ϕ_i are Euler angles and M_i, M_j, M_k are rotation matrices.

It will now be shown that the matrix V (y) of (51) is skew-symmetric.

Consider

$$\vec{y}\rangle \cdot \langle\bar{y} = I$$

and differentiating

$$\dot{\vec{y}}\rangle \cdot \langle\bar{y} + \vec{y}\rangle \cdot \langle\dot{\bar{y}} = [0]$$

hence

$$\dot{\vec{y}}\rangle \cdot \langle\bar{y} = - \vec{y}\rangle \cdot \langle\dot{\bar{y}} \tag{75}$$

20

Transposing (51)

$$\dot{\overrightarrow{y}}\rangle = V'(y)\, \overrightarrow{y}\rangle \tag{76}$$

projecting (76) onto $\langle\overrightarrow{y}$, that is

$$\overrightarrow{y}\rangle \cdot \langle\overrightarrow{y} = V'(y)\, \overrightarrow{y}\rangle \cdot \langle\overrightarrow{y} = V'(y) \tag{77}$$

and projecting (51) onto $\overrightarrow{y}\rangle$ that is

$$\overrightarrow{y}\rangle \cdot \langle\dot{\overrightarrow{y}} = V(y) \tag{78}$$

using (77) and (78) in (75)

$$V'(y) = -V(y)$$

or

$$V(y) = -V'(y) \tag{79}$$

hence the matrix V is skew-symmetric. Define its elements as

$$V = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{80}$$

The relationship to Euler angles is given in reference (4)

The matrix equation (63) becomes

$$\begin{pmatrix} 0 & \dot{\omega}_3 & -\dot{\omega}_2 \\ -\dot{\omega}_3 & 0 & \dot{\omega}_1 \\ \dot{\omega}_2 & -\dot{\omega}_1 & 0 \end{pmatrix} + \begin{pmatrix} -\left(\omega_2^2 + \omega_3^2\right) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -\left(\omega_1^2 + \omega_3^2\right) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -\left(\omega_2^2 + \omega_1^2\right) \end{pmatrix} = H \tag{81}$$

21

Equating the indicated elements of Equation (81)

$$\dot{\omega}_1 + \omega_3\,\omega_2 = h_{23}$$

$$\dot{\omega}_2 + \omega_3\,\omega_1 = h_{31} \qquad\qquad (82)$$

$$\dot{\omega}_3 + \omega_1\,\omega_2 = h_{12}$$

Equation (82) may be put into matrix form as

$$\dot{\omega}\Big\rangle + \begin{pmatrix} 0 & \omega_3 & 0 \\ \omega_3 & 0 & 0 \\ 0 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} h_{23} \\ h_{31} \\ h_{12} \end{pmatrix} \qquad (83)$$

The above non-linear transformation matrix is singular (non-invertible) and is not unique, for example

$$\begin{pmatrix} \omega_3\,\omega_2 \\ \omega_3\,\omega_1 \\ \omega_1\,\omega_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \omega_2 \\ 0 & 0 & \omega_1 \\ 0 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad (84)$$

Observe that the transformation matrix of Equation (83) is a function of ω_1 and ω_3 whereas the matrix of (84) is a function of ω_1 and ω_2.

If we make (83) look like a vector in the $\langle \bar{y}$ space

$$\langle \bar{y}\;\dot{\omega}\rangle + \langle \bar{y}\; T\,(\omega)\rangle\;\omega\rangle = \langle \bar{y} \begin{pmatrix} h_{23} \\ h_{31} \\ h_{12} \end{pmatrix}$$

22

then the vector components are

$$h_{23} = \bar{y}_2 \cdot \bar{h}_3$$

$$h_{31} = \bar{y}_3 \cdot \bar{h}_1$$

$$h_{12} = \bar{y}_1 \cdot \bar{h}_2$$

and

$$\bar{\tau} = \left\langle \bar{y}\, h \right\rangle = \left\langle \bar{y} \begin{pmatrix} \bar{y}_2 \cdot \bar{h}_3 \\ \bar{y}_3 \cdot \bar{h}_1 \\ \bar{y}_1 \cdot \bar{h}_2 \end{pmatrix} \right.$$

$$\bar{\tau} = \left(\bar{y}_1\, \bar{y}_2 \right) \bar{h}_3 + \left(\bar{y}_2\, \bar{y}_3 \right) \bar{h}_1 + \left(\bar{y}_3\, \bar{y}_1 \right) \bar{h}_2$$

Clearly (63) requires only three integrations rather than nine as before.
Hence the system (65) reduces to 18 integrators. However, these are not
linearly independent differential equations as is known, only twelve are
linearly independent. Let us find the redundant ones.

Consider Equation (64), that is

$$V(\omega) = Y^{-1}\, \dot{Y}$$

Equating the three unique elements of V in (85) to the right hand
matrix product yields

$$\omega_1 = \omega_1 \left(\omega, \dot{\omega} \right)$$

$$\omega_2 = \omega_2 \left(\omega, \dot{\omega} \right) \tag{85}$$

$$\omega_3 = \omega_3 \left(\omega, \dot{\omega} \right)$$

23

The relations (85) are well known to imply

$$\omega\rangle = T\left(\dot{\phi}\rangle\right)\dot{\phi}\rangle$$

where $T\left(\phi\rangle\right)$ is invertible so long as the second angle is not 90°, that is

$$\dot{\phi}\rangle = T^{-1}\,\omega\rangle\ .\tag{86}$$

One can now use three integrations to obtain $\phi\rangle$ and then generate the trigonometric relations of the matrix Y since

$$Y = Y\left(\phi\rangle\right)$$

The constrained system now has twelve integrations

$$\boxed{\begin{aligned}
&\ddot{xy}\rangle + 2\,V(\omega\rangle)\,\dot{xy}\rangle + \left(\dot{V}(\omega) + V^2\right)xy\rangle = a\rangle\\
&\dot{\omega}\rangle + T(\omega\rangle)\,\omega\rangle = h\rangle\\
&\dot{\phi}\rangle = T^{-1}(\phi)\,\omega\rangle
\end{aligned}}\tag{87}$$

The above system of equations looks like the standard six degree of freedom rigid-body dynamics. Clearly twelve integrations are required.

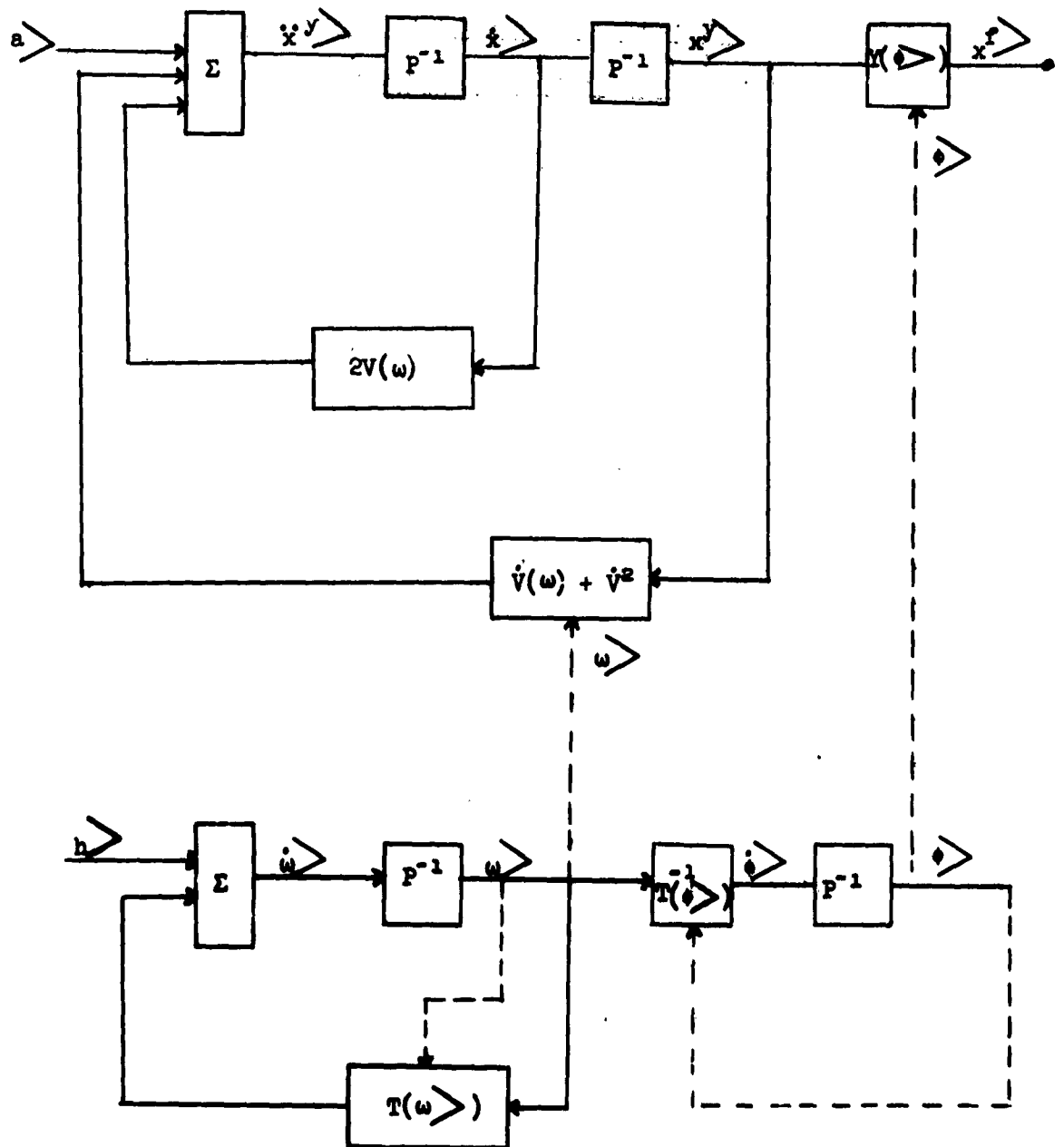The transfer-matrix flow diagram of the system (87) is shown in Figure (1).

FIGURE (1) SIX DEGREES OF FREEDOM SYSTEM

Section IV, <u>The Trajectory of a Vector $\bar{x}$ and two Arbitrary Time-Varying</u>

<u>Bases $\langle\bar{y}$ and $\langle\bar{z}$ (42 Integrations).</u>

In aerospace vehicle math-modelling one must consider many time varying bases. This section considers the system of vectors

$$\left(\bar{x}, \langle\bar{y}, \langle\bar{z} \right) \tag{1}$$

and the relations

$$\bar{x} = \left\langle \bar{f}\ x^f \right\rangle = \left\langle \bar{y}\ x^y \right\rangle = \left\langle \bar{z}\ x^f \right\rangle \tag{2}$$

where

$$\langle\bar{z} = \langle\bar{f}\ Z \tag{3}$$

or by

$$\langle\bar{z} = \langle\bar{y}\ Y^{-1}\ Z = \langle\bar{y}\ T\ . \tag{4}$$

$$Z = Y\ T \tag{5}$$

By analogy with (II-60)

$$\langle\overset{..}{\bar{z}} = \langle\bar{z}\ \left[V^2(z) + \dot{V}(z) \right] = \langle\bar{z}\ H(z) \tag{6}$$

26

The system equations are by (II-65) (II-17) and 5

$$\ddot{x^y}\rangle + 2\ V(y)\ \dot{x^y}\rangle + \left[V^2(y) + \dot{V}(y)\right] x^y\rangle = a^y\rangle$$

6 integrators

$$\dot{V}(y) + V^2(y) = H(y)$$

9 integrators

$$\dot{V}(z) + V^2(z) = H(z)$$

9 integrators

$$\dot{T} + V(y)\ T - T\ V(z) = [0] \qquad\qquad (7)$$

9 integrators

$$\dot{Y} = Y\ V(y)$$

9 integrators

Algebraic Relations

$$Z = Y\ T$$

$$x^f\rangle = Y\ x^y\rangle$$

and

$$x^z\rangle = T^{-1}\ x^y\rangle$$

The above system requires 42 integrations plus the indicated algebraic manipulations. The system is not unique, that is one could obtain many other math models of the same dynamical system. For example, consider

$$\langle \bar{z} = \langle \bar{y}\ T$$

27

and

$$\left\langle \dot{\vec{z}} \right. = \left\langle \dot{\vec{y}} \right. T + \left\langle \vec{y} \right. \dot{T}$$

and

$$\left\langle \ddot{\vec{z}} \right. = \left\langle \ddot{\vec{y}} \right. T + 2 \left\langle \dot{\vec{y}} \right. \dot{T} + \left\langle \vec{y} \right. \ddot{T} = \left\langle \vec{z} \right. H(z)$$

by (II-3) and (II-60)

$$\left\langle \ddot{\vec{z}} \right. = \left\langle \vec{y} \right. \left[\dot{V}(y) + 2 \, V^2(y) \right] T + 2 \left\langle \vec{y} \right. V(y) \, T + \left\langle \vec{y} \right. \ddot{T}$$

or

$$\ddot{T} + 2 \, V(y) \, \dot{T} + \left[\dot{V}(y) + 2 \, V^2(y) \right] T = H(z) \quad . \qquad (8)$$

Hence the system is defined by

$$\boxed{\begin{array}{l}
\ddot{x}\vec{y}\!\!\bigg\rangle + 2 \, V(y) \, \dot{x}\vec{y}\!\!\bigg\rangle + \left[\dot{V}(y) + V^2(y) \right] x\vec{y}\!\!\bigg\rangle = a\vec{y}\!\!\bigg\rangle \\
\hfill \text{6 integrators} \\[2mm]
\dot{V}(y) + V^2(y) = H(y) \\
\hfill \text{9 integrators} \\[2mm]
\ddot{T}(y) + 2 \, V(y) \, \dot{T} + \left[\dot{V} + V^2(y) \right] T = [0] \qquad (9)\\
\hfill \text{18 integrators} \\[2mm]
\dot{Y} = Y \, V(y) \\
\hfill \text{9 integrators} \\[2mm]
Z = Y \, T
\end{array}}$$

When the two moving bases $\left\langle \vec{y} \right.$ and $\left\langle \vec{z} \right.$ are each orthonormal bases and the transformation matrix T is a function of only three parameters - namely the Euler angles orienting the two bases then one reduces the system (9) to a system of 9 degrees of freedom or 18 integrations. The details of

the derivation are not presented in this paper. The classical approach
to this case introduces such "poor" notions as "pseudo vector" or axial
vectors and relations like

$$\bar{\omega}_z = \bar{\omega}_{zy} + \bar{\omega}_y$$

where the "axial" angular velocity vectors $\bar{\omega}_z$ and $\bar{\omega}_y$ are "inertial angular
velocities" and $\bar{\omega}_{zy}$ is the relative angular velocity vector.

REFERENCES

1. Ince, E. L., *Ordinary Differential Equations*, Dover Publications, 1926, Page 23.

2. Levine, J. J., "On the Matrix Riccati Equation," Proceedings of the American Mathematical Society, Volume 10, 1959.

3. Halmos, Paul R., *Finite Dimensional Vector Spaces*, D. Van Nostrand Co, Inc., Princeton, New Jersey, 1958, Page 86.

4. Pappas, J. S., "Derivation of the Euler Angle Rate Equations Using Rotations Axes or a Non-Orthogonal Bases - With Tables of Matrix Equations for the Rotation Sequences," Technical Note Nr. 9, Electro-Mechanical Laboratory, Army Missile Test and Evaluation Directorate, White Sands Missile Range, May 1963.